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Hybrid Laplace transform technique for non-linear transient thermal problems

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Abstract—A hybrid numerical method combining the application of the Laplace transform technique and the finite-difference method (FDM) or the finite-element method (FEM) is presented for non-linear transient thermal problems. The space domain in the governing equation is discretized by FDM or FEM and the non-linear terms are linearized by Taylor's series expansion. The time-dependent terms are removed from the linearized equations by Laplace transformation, and so, the results at a specific time can be calculated without step-by-step computation in the time domain. To show the efficiency and accuracy of the present method several one-dimensional non-linear transient thermal problems are studied.

INTRODUCTION

It is of great importance in most engineering and science applications to analyse the transient thermal response within a solid with temperature-dependent thermal properties or within a structure subjected to a radiative boundary condition. The governing equations with either the temperature-dependent thermal properties or the radiative boundary condition will become nonlinear. Owing to no general mathematic theory to analytically solve non-linear partial differential equations, various approximate methods [1–6] have been published to find approximate solutions. To find a more accurate solution, numerical methods should be employed. Up to date, the finite-element method (FEM), the finite-difference method (FDM), and the boundary-element method (BEM) are commonly used. In the application of those numerical methods to transient problems, the time derivative is often dealt with by the modified Crank–Nicolson method [7], the time-integration method [8] or other methods [9, 10]. However, the major drawback of these methods is that the calculation must be performed at each time step until the specific time is reached and it will cost a great deal of computer time to obtain a long-time solution for these methods.

In the present study, the hybrid application of the Laplace transform method and the FEM (or the FDM) is used to analyse the non-linear transient heat conduction through a solid. This hybrid method has been proved to be very powerful for linear transient problems [11, 12]. A few works have been carried out on the method of the Laplace transform for non-linear transient heat conduction problems. Tamma and Railkar [13–15] applied the hybrid transfinite element methodology to solve non-linear transient problems. Based on the assumptions that (1) the thermal properties are constant within an element, and (2) the element thermal properties depend only on the average element temperatures, the thermal-equilibrium equations in the transformed domain obtained by them are nonlinear. However, these assumptions are reasonable for lower order finite elements with simple temperature variations [16]. Based on this reason, the present study applies Taylor's series expansion to linearize the non-linear terms. It is seen that the thermal-equilibrium equations in the transformed domain are not nonlinear, but linear. Owing to no time step in the present hybrid method, the results at a specific time can be calculated without iteration at each time step. To show the efficiency and accuracy of this hybrid method for such problems, three one-dimensional test examples consisting of (1) a hollow sphere subjected to a convective–radiative boundary condition, (2) a slab with temperature-dependent thermal conductivity and specific heat, and (3) a conductive–convective–radiative fin with temperature-dependent thermal properties are analysed. A comparison of the hybrid finite-element method solutions (FES) and the hybrid finite-difference solutions (FDS) with the results obtained by a conventional finite-difference method using the Crank–Nicolson algorithm is made. It can be seen that there is no remarkable difference between them.

DESCRIPTION OF THE PROPOSED HYBRID SCHEME

Consider a general one-dimensional transient non-linear thermal problem described by the following differential equation:

$$\mathbf{D}(T, x, t) = 0 \quad \text{in } \Omega \tag{1}$$

with the boundary condition

$$\mathbf{B}(T, x, t) = 0 \quad \text{on } \mathbb{R} \tag{2}$$

and the initial condition

$$\mathbf{I}(T, x, 0) = 0 \tag{3}$$

where **D** and **B** are non-linear differential operators, Ω the domain of the problem and $\mathbb{R}$ its boundary.

The application of the present hybrid numerical

NOMENCLATURE

A	cross-sectional area of the fin
$\mathbf{B}$	boundary differential operator
Bi	Biot number
b	thickness of the slab
$\mathscr{C}(\theta)$	dimensionless specific heat
C_{p0}	reference specific heat at $T = 0$
$\mathbf{D}$	transient conduction differential operator
$\mathbf{D}^*$	non-linear algebraic equation
$\bar{\mathbf{D}}$	linearized algebraic equation
$\tilde{\mathbf{D}}$	Laplace transformed algebraic equation
$\{f\}$	global force vector
h	heat transfer coefficient
$\mathbf{I}$	initial condition
$\mathscr{K}(\theta)$	dimensionless thermal conductivity
$[K]$	global conduction matrix
$k(T)$	thermal conductivity
k_0	reference thermal conductivity at $T = 0$
L	length of the fin
l	distance between two nodes
N	pyramid function
n	total number of nodes
P	perimeter of the fin cross-section
p	parameter in the free convective heat transfer coefficient
R	real domain or $= R_o - R_i$
R_i	inner radius of the hollow sphere
R_o	outer radius of the hollow sphere
r	radius of the sphere
$\mathscr{r}$	dimensionless radius
s	Laplace transform parameter
T	temperature
$\bar{T}$	previous iterative temperature
$\tilde{T}$	Laplace transformed temperature
$\{T\}$	global temperature vector
t	time
t_s	specific time
v, w	free parameters
x	space coordinate
$\mathscr{x}$	dimensionless space coordinate.

Greek symbols

β	temperature coefficient for the thermal conductivity
γ	temperature coefficient for the specific heat
ε	emissivity
θ	dimensionless temperature
κ	constant in equation (26)
λ	constant in equation (27)
ξ	fin parameter
ρ	density
σ	Stefan–Boltzmann constant
τ	dimensionless time
ω	reference thermal diffusivity ratio, α_{01}/α_{02}
Ω	domain of the problem.

Other symbols

$\mathbb{R}$	boundary of the problem
$\exists$	tolerance error.

Subscripts

∞,a	environment
b	fin base
e	effective sink
i	ith node
j	jth node
w	wall.

method to find the solution of equations (1)–(3) can be divided into the following steps.

(1) Discretize the governing equation

Equations (1) and (2) can be discretized either by the Galerkin finite-element method which approximates the temperature distribution T on the interval $[x_{j-1}, x_{j+1}]$ as

$$T(x,t) = N_{j-1}(x)T_{j-1} + N_j(x)T_j + N_{j+1}(x)T_{j+1} \tag{4}$$

where $N_k(x)$, $k = j-1, j, j+1$ are the pyramid functions defined by

$$N_k(x) = \begin{cases} \dfrac{x - x_{k-1}}{l} & x \in [x_{k-1}, x_k] \\ \dfrac{x_{k+1} - x}{l} & x \in [x_k, x_{k+1}] \\ 0 & \text{elsewhere} \end{cases} \tag{5}$$

and l denotes the distance between two nodes and gives the discretized heat conduction equation in the form

$$\int_\Omega \mathbf{D}(T, x, t) N_j \, \mathrm{d}x = 0 \tag{6}$$

or by the finite-difference method. After discretizing the governing equations, the following set of non-linear algebraic equations can be obtained:

$$\mathbf{D}^*(T_d, x_j, t) = 0 \quad j = 1, 2, \ldots, n-1, n \tag{7}$$

where n is the total number of nodes and d is defined as

$$d = \begin{cases} 1, 2 & \text{when } j = 1 \\ j-1, j, j+1 & \text{when } j = 2, 3, \ldots, n-1 \\ n-1, n & \text{when } j = n \end{cases} \tag{8}$$

(2) Linearize the non-linear terms

The application of the Laplace transform technique is only restricted to the linear system, so that the non-linear terms in equation (7) must be linearized. In the present study, Taylor's series expansion is applied to linearize the non-linear terms. Let $f(\eta_1, \eta_2, \ldots, \eta_m)$ be a many-times differentiable non-linear function of $\eta_1, \eta_2, \ldots$, and η_m. Then its Taylor's series expansion is given as

$$f(\eta_1, \eta_2, \ldots, \eta_m) = f(\bar{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_m) + \sum_{i=1}^{m} \frac{\partial}{\partial \eta_i} f(\bar{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_m)(\eta_i - \bar{\eta}_i) \quad (9)$$

where the overbar denotes the previously iterated solution. Substituting equation (9) into equation (7) can yield the following linear algebraic equations:

$$\bar{\mathbf{D}}(T_d, \bar{T}_d, x_j, t) = 0. \quad (10)$$

(3) Remove the time-dependent terms

In order to remove the time-dependent terms from equation (10), the Laplace transform technique is utilized. The Laplace transform of a real function $g(t)$ and its inversion formula are defined as

$$\tilde{g}(s) = \mathscr{L}[g(t)] = \int_0^{\infty} e^{-st} g(t)\, dt \quad (11)$$

and

$$g(t) = \mathscr{L}^{-1}[\tilde{g}(s)] = \frac{1}{2\pi i} \int_{v-i\infty}^{v+i\infty} e^{st} \tilde{g}(s)\, ds \quad (12)$$

where $s = v + iw$, v, $w \in R$.

The transformed form of equation (10) can be written as

$$\tilde{\mathbf{D}}(\tilde{T}_d, \bar{T}_d, x_j, s) = 0. \quad (13)$$

(4) Solve the transformed equation

Equation (13) is a vector-matrix form equation of the following type:

$$[K]\{\tilde{T}\} = \{f\} \quad (14)$$

where $[K]$ is an $(n \times n)$ band matrix with complex numbers, $\{\tilde{T}\}$ an $(n \times 1)$ vector representing the transformed nodal temperature and $\{f\}$ an $\{n \times 1\}$ vector representing the forcing terms. Note that equation (14) is a linear equation. An initial temperature distribution $\bar{T}(x, t_s)$ at the specific time t_s is guessed and then $[K]$ and $\{f\}$ can be calculated. The double direct Gaussian-elimination algorithm is used to solve $\{\tilde{T}\}$ and the numerical inversion of the Laplace transform technique [17–19] is applied to invert the transformed result to the physical quantity $\{T\}$. These updated values of $\{T\}$ are used to calculate $\{K\}$ and $\{f\}$ for iteration. This computational procedure is performed repeatedly until a desired convergence is achieved. Equation (14) derived by Tamma and Railkar [13–15] is nonlinear. They solved the system of non-linear equations using the Newton–Raphson method. According to the comment of Shih [20], the Newton–Raphson method converges rapidly if the initial guess lies within the vicinity of the solution. It will diverge, however, if the starting solution cannot be well guessed.

An iterative solution is said to be convergent if

$$\frac{|T_j^k - T_j^{k+1}|}{T_j^k} < \exists \quad \text{as } j = 1, 2, \ldots, n \quad (15)$$

where the superscript k denotes the kth iteration and $\exists$ the tolerance error. In the present study, the first iteration values of T_j are set to zero and the tolerance error $\exists$ is chosen as 10^{-4}.

ILLUSTRATIVE EXAMPLES

In the following, three kinds of examples are studied to illustrate the applicability of the present hybrid numerical method to non-linear transient thermal problems. In each example, the 11-node modelling is used for both FDM and FEM and the numerical inversion of the Laplace transform proposed by Honig and Hirdes [19] is employed. The comparative results are obtained by the conventional finite-difference method using the Crank–Nicolson algorithm. To obtain more accurate comparative results, 21 nodes and a time step $\Delta t = 0.01$ are chosen. All the computation is performed on a PC with an 80386 microprocessor and the program is written in FORTRAN.

Example 1: a hollow sphere with a convective–radiative boundary condition

In some engineering applications, hollow-spherical structures such as reservoirs are used and a convective–radiative heat transfer usually occurs between the structure surface and the environment. This example studies a hollow sphere with constant thermal properties which is subjected to a convective–radiative boundary condition at the outer surface. The dimensionless form of the governing equations describing the transient thermal response of the sphere is given by

$$\frac{\partial \theta}{\partial \tau} = \frac{1}{\imath^2} \frac{\partial}{\partial \imath}\left(\imath^2 \frac{\partial \theta}{\partial \imath}\right) \quad (16a)$$

$$\theta = 0 \quad \text{at } \imath = \mathscr{R}_i \quad (16b)$$

$$\frac{\partial \theta}{\partial \imath} = \omega(1 - \theta^4) + Bi(1 - \theta) \quad \text{at } \imath = \mathscr{R}_o \quad (16c)$$

$$\theta = 0 \quad \text{at } \tau = 0 \quad (16d)$$

where $\theta = T/T_{\infty}$, $\imath = r/(R_o - R_i) = r/R$, $\tau = k_0 t/R^2 \rho_0 C_{p0}$, $\mathscr{R}_i = R_i/R$, $\mathscr{R}_o = R_o/R$, $\omega = R\varepsilon\sigma T_{\infty}^3/k_0$, $Bi = Rh/k_0$, R_i is the inner radius, R_o the outer radius, and T_{∞} the environment temperature. Equations (16) can be solved by the scheme described in the above section and the non-linear term shown in equation (16c) is linearized as

$$\theta_n^4 = 4\bar{\theta}_n^3 \theta_n - 3\bar{\theta}_n^4 \quad (17)$$

where $\bar{\theta}_n$ denotes the previously iterated temperature

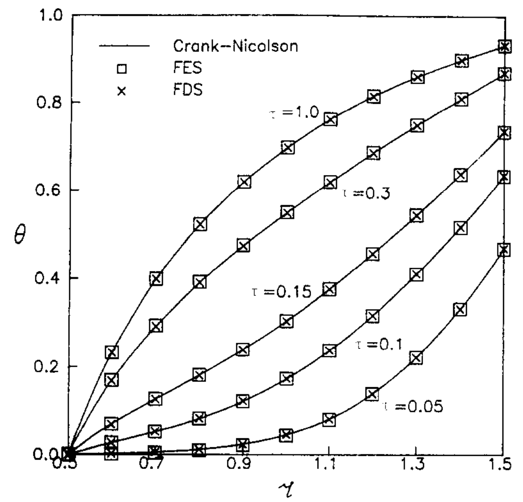

FIG. 1. A comparison of the temperature distribution for various τ.

at $r = \mathscr{R}_o$. Figure 1 shows a comparison of the present hybrid numerical solutions with the results given by the finite-difference method using the Crank–Nicolson algorithm for the case $\mathscr{R}_i = 0.5$, $\mathscr{R}_o = 1.5$, $\omega = 1$, and $Bi = 1$. It can be seen that there is no remarkable difference between them. Moreover, the iteration is needed only at the outer-surface temperature at the specific time. The iterative procedures are finished until the desired convergence is reached. However, the application of the Crank–Nicolson algorithm must iterate all the nodal values within the solid at each time step. In the present computation, four or five iterations are needed. The CPU time (PC with 80386 microprocessor) takes about 5 s to obtain a convergent solution for any specific dimensionless time.

Example 2: a slab with temperature-dependent thermal properties

This example considers a single slab with temperature-dependent thermal conductivity and specific heat. The dimensionless form of the governing equations are given as

$$\mathscr{C}(\theta)\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left(\mathscr{K}(\theta)\frac{\partial \theta}{\partial x}\right) \quad (18a)$$

$$\theta = 1 \quad \text{at } x = 0 \quad (18b)$$

$$\theta = 0 \quad \text{at } x = 1 \quad (18c)$$

$$\theta = 0 \quad \text{at } \tau = 0 \quad (18d)$$

where $\theta = T/T_w$, $x = x/b$, $\tau = k_0 t/b^2\rho_0 C_{p0}$, b is the thickness of the slab, and T_w the wall temperature. The linear variation of the thermal conductivity and specific heat with temperature is assumed and they can be expressed as

$$\mathscr{C}(\theta) = 1+\gamma\theta \quad \text{and} \quad \mathscr{K}(\theta) = 1+\beta\theta. \quad (19)$$

Applying the FDM to the right-hand side of equation (18a) gives

$$\frac{\partial}{\partial x}\left[(1+\beta\theta)\frac{\partial \theta}{\partial x}\right] = \frac{1}{l^2}(\theta_{i-1}-2\theta_i+\theta_{i+1}) + \frac{\beta}{2l^2}(\theta_{i-1}^2-2\theta_i^2+\theta_{i+1}^2). \quad (20)$$

The non-linear terms θ^2 can be linearized, using equation (9), into the following form:

$$\theta^2 = 2\bar{\theta}\theta-\bar{\theta}^2. \quad (21)$$

The left-hand side of equation (18a) can be linearized by the following process:

$$\mathscr{C}(\theta)\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial \tau}\left[\int_0^\theta \mathscr{C}(\eta)\,\mathrm{d}\eta\right] = \frac{\partial}{\partial \tau}[E(\theta)] \quad (22)$$

where $E(\theta)$ is a non-linear function. Then $E(\theta)$ can be linearized, using equation (9), into the following form:

$$E(\theta) = E(\bar{\theta})+\left[\frac{\partial}{\partial \theta}E(\theta)\right]_{\theta=\bar{\theta}}(\theta-\bar{\theta}). \quad (23)$$

Substituting equation (23) into equation (22) yields the linearized form as

$$\mathscr{C}(\theta)\frac{\partial \theta}{\partial \tau} = \mathscr{C}(\bar{\theta})\frac{\partial \theta}{\partial \tau}. \quad (24)$$

Thus, by the present hybrid FDM, equation (18a) can be written as

$$\left[\frac{1}{l^2}(1+\beta\bar{\theta}_{i-1})\right]\tilde{\theta}_{i-1}+\left[\frac{-2}{l^2}(1+\beta\bar{\theta}_i)-(1+\gamma\bar{\theta}_i)s\right]\tilde{\theta}_i+\left[\frac{1}{l^2}(1+\beta\bar{\theta}_{i+1})\right]\tilde{\theta}_{i+1} = \frac{\beta}{2l^2s}(\bar{\theta}_{i-1}^2-2\bar{\theta}_i^2+\bar{\theta}_{i+1}^2). \quad (25)$$

When the hybrid FEM is used, equation (18a) is expressed as

$$\left[\frac{1}{l^2}(1+\beta\bar{\theta}_{i-1})-\frac{s}{6}(1+\gamma\bar{\theta}_i)\right]\tilde{\theta}_{i-1}+\left[\frac{-2}{l^2}(1+\beta\bar{\theta}_i) - \frac{2s}{3}(1+\gamma\bar{\theta}_i)\right]\tilde{\theta}_i+\left[\frac{1}{l^2}(1+\beta\bar{\theta}_{i+1})-\frac{s}{6}(1+\gamma\bar{\theta}_i)\right]\tilde{\theta}_{i+1} = \frac{\beta}{2l^2s}(\bar{\theta}_{i-1}^2-2\bar{\theta}_i^2+\bar{\theta}_{i+1}^2). \quad (26)$$

The comparison of the present numerical solutions with the results given by the finite-difference method using the Crank–Nicolson algorithm is shown in

Table 1. A comparison of the results for $\gamma = 0.5$, $\beta = 1.0$ and various τ

	$\tau = 0.1$			$\tau = 1.0$		
x	FDM	FEM	Crank–Nicolson	FDM	FEM	Crank–Nicolson
0.2	0.7341	0.7348	0.7337	0.8439	0.8439	0.8439
0.4	0.4710	0.4718	0.4705	0.6733	0.6733	0.6733
0.6	0.2510	0.2509	0.2513	0.4832	0.4832	0.4832
0.8	0.0993	0.0986	0.1003	0.2649	0.2649	0.2649

Table 2. A comparison of the results for $\tau = 0.5$, $\beta = 0.5$ and various γ

	$\gamma = -0.5$			$\gamma = 1.0$		
x	FDM	FEM	Crank–Nicolson	FDM	FEM	Crank–Nicolson
0.2	0.8284	0.8283	0.8283	0.8216	0.8219	0.8230
0.4	0.6456	0.6456	0.6455	0.6341	0.6347	0.6365
0.6	0.4494	0.4494	0.4492	0.4372	0.4377	0.4397
0.8	0.2360	0.2360	0.2359	0.2279	0.2283	0.2296

Table 3. A comparison of the results for $\tau = 0.5$, $\gamma = 0.5$ and various β

	$\beta = -0.5$			$\beta = 1.0$		
x	FDM	FEM	Crank–Nicolson	FDM	FEM	Crank–Nicolson
0.2	0.7149	0.7156	0.7192	0.8430	0.8431	0.8430
0.4	0.4910	0.4918	0.4970	0.6717	0.6718	0.6718
0.6	0.3055	0.3060	0.3101	0.4813	0.4815	0.4815
0.8	0.1448	0.1451	0.1469	0.2635	0.2636	0.2636

Tables 1–3. It is seen that they are in good agreement with each other. Four or five iterations are required to obtain a convergent solution. The CPU time (PC with 80386 microprocessor) needs about 15 s for any specific dimensionless time.

Example 3: a conductive–convective–radiative fin

This example considers a horizontal straight fin of length **L**, cross-sectional area **A**, and perimeter **P** exposed to simultaneously free convective and radiative heat transfer on the surface. The boundary conditions of constant base temperature T_b and adiabatic tip are assumed. The convective environment temperature and effective sink temperature are T_a and T_e, respectively. The density ρ, specific heat C_p, and surface emissivity ε are taken to be constant while the thermal conductivity is assumed to be of the form

$$k(T) = k_0[1+\kappa(T-T_a)] \tag{27}$$

and the convective heat transfer coefficient h is assumed as

$$h = \lambda(T-T_a)^p \tag{28}$$

where λ is a constant and p a small parameter (p = 0.25 and 0.33 for laminar and turbulent conditions, respectively [4]). The dimensionless governing equations for this example may be written as

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial}{\partial x}\left[(1+\beta(\theta-\theta_a))\frac{\partial\theta}{\partial x}\right] - \xi^2(\theta-\theta_a)^{1+p} - \omega(\theta^4-\theta_e^4) \tag{29a}$$

$$\theta = 1 \quad \text{at } x = 0 \tag{29b}$$

$$\frac{\partial\theta}{\partial x} = 0 \quad \text{at } x = 1 \tag{29c}$$

$$\theta = 0 \quad \text{for } \tau = 0 \tag{29d}$$

where $\theta = T/T_b$, $\theta_a = T_a/T_b$, $\theta_e = T_e/T_b$, $x = x/L$, $\tau = k_0 t/L^2\rho_0 C_{p0}$, $\beta = \kappa T_b$, $\xi^2 = h_b PL^2/k_0 A$, $\omega = \varepsilon\sigma T_b^3 PL^2/k_0 A$, $h_b = \lambda(T_b-T_a)^p$. In the following computation, the case $\theta_a = \theta_e = 0$ is considered.

The application of the FDM to discretize equation (29a) is straightforward and gives

$$\frac{\partial\theta}{\partial\tau} = \frac{1}{l^2}(\theta_{i-1}-2\theta_i+\theta_{i+1}) + \frac{\beta}{2l^2}(\theta_{i-1}^2-2\theta_i^2+\theta_{i+1}^2) - \xi^2\theta_i^{1+p} - \omega\theta_i^4. \tag{30}$$

Linearizing the non-linear terms in equation (30) by using equation (9) and then taking the Laplace transform of the linearized equation can yield the following equation:

$$[1+\beta\bar{\theta}_{i-1}]\tilde{\theta}_{i-1} + [-2-2\beta\bar{\theta}_i - l^2(\xi^2(1+p)\bar{\theta}_i^p - 4\omega\bar{\theta}_i^3 - s)]\tilde{\theta}_i + [1+\beta\bar{\theta}_{i+1}]\tilde{\theta}_{i+1} = \frac{\beta}{2s}(\bar{\theta}_{i-1}^2-2\bar{\theta}_i^2+\bar{\theta}_{i+1}^2) - \frac{l^2}{s}(\xi^2 p\bar{\theta}_i^{1+p}+3\omega\bar{\theta}_i^4). \tag{31}$$

However, the application of the FEM, using equations (4)–(6), is difficult to discretize the convective and radiative terms in equation (29a). Thus, before the discretizing procedure, the non-linear terms due to convection and radiation in the ith element are firstly linearized into the following forms:

$$\theta^{1+p} = -p\bar{\theta}_i^{1+p} + (1+p)\bar{\theta}_i^p\theta \tag{32}$$

and

$$\theta^4 = -3\bar{\theta}_i^4 + 4\bar{\theta}_i^3\theta \tag{33}$$

where $\bar{\theta}_i$ is the previously iterated ith node temperature, and the discretized and transformed form of equation (29a) given by the hybrid finite-element method is written as

$$\left[1+\beta\bar{\theta}_{i-1} - \frac{l^2}{6}(s+\xi^2(1+p)\bar{\theta}_i^p+4\omega\bar{\theta}_i^3)\right]\tilde{\theta}_{i-1} + \left[-2-2\beta\bar{\theta}_i - \frac{2l^2}{3}(s+\xi^2(1+p)\bar{\theta}_i^p+4\omega\bar{\theta}_i^3)\right]\tilde{\theta}_i + \left[1+\beta\bar{\theta}_{i+1} - \frac{l^2}{6}(s+\xi^2(1+p)\bar{\theta}_i^p+4\omega\bar{\theta}_i^3)\right]\tilde{\theta}_{i+1} = \frac{\beta}{2s}(\bar{\theta}_{i-1}^2-2\bar{\theta}_i^2+\bar{\theta}_{i+1}^2) - \frac{l^2}{s}(\xi^2 p\bar{\theta}_i^{1+p}+3\omega\bar{\theta}_i^4). \tag{34}$$

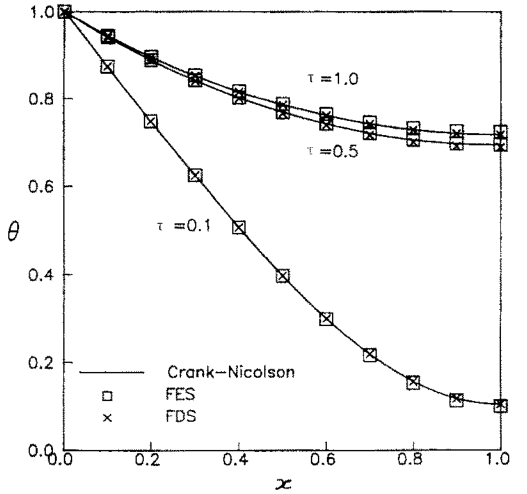

FIG. 2. A comparison of the temperature distribution for $\beta = 1$, $\xi = 1$, $p = 0.33$, $\omega = 1$ and various τ.

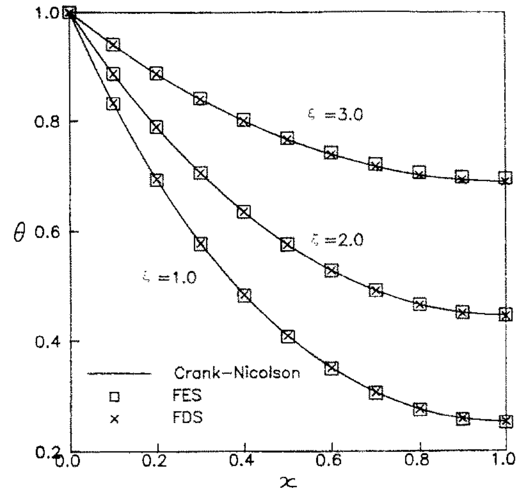

FIG. 4. A comparison of the temperature distribution for $\beta = 1$, $p = 0.33$, $\omega = 1$, $\tau = 0.5$ and various ξ.

Figures 2–5 show the comparison of the temperature distribution within the fin given by the present hybrid numerical schemes and by the conventional FDM using the Crank–Nicolson algorithm. It can be seen from these figures that the present hybrid numerical solutions agree well with the results using the Crank–Nicolson algorithm. For the present hybrid methods, four or five iterations are needed to obtain a convergent solution at a specific dimensionless time. The CPU time (PC with 80386 microprocessor) is taken as about 15 s.

CONCLUSION

The present study applies the hybrid numerical method involving the Laplace transform technique and the FEM (or the FDM) to non-linear transient thermal problems. From the illustrated examples, it can be seen that the proposed hybrid numerical method is efficient and accurate to determine the non-linear transient thermal response within a solid.

To the authors' knowledge, the solution of n simultaneous algebraic equations at each time step is required when the Crank–Nicolson algorithm is used, i.e. all the internal temperatures must be calculated at each time step. This procedure will tend to increase the cost when the solutions must be carried out over long-time periods. Furthermore, it is often necessary to take very small time steps to avoid undesirable numerical oscillations in the solution. This severe limitation on the time step may require an excessive amount of computer time. The advantage of the application of the Laplace transform technique in the present method is that it can quickly give an accurate solution at a specific time without step-by-step computation in the time domain. This advantage is

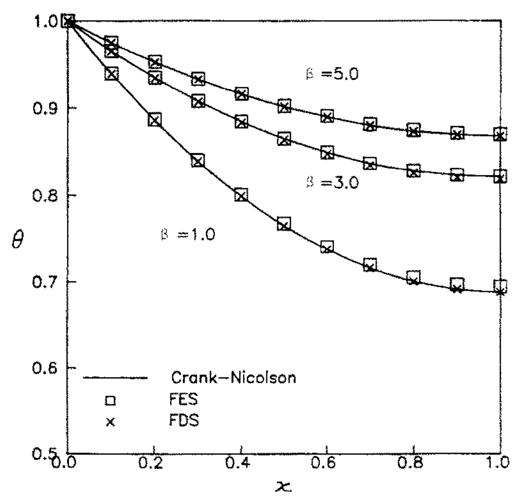

FIG. 3. A comparison of the temperature distribution for $\xi = 1$, $p = 0.33$, $\omega = 1$, $\tau = 0.5$ and various β.

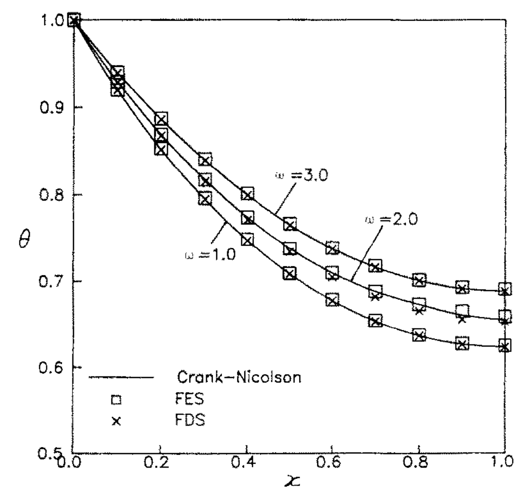

FIG. 5. A comparison of the temperature distribution for $\beta = 1$, $\xi = 1$, $p = 0.33$, $\tau = 0.5$ and various ω.

especially powerful when a long-time solution is required.

The present study only gives an indication of the basic procedure of the proposed hybrid numerical scheme. The procedure as described in the present study should be applicable to most non-linear transient problems. The further application of the present hybrid numerical method to other non-linear transient problems, such as Navier–Stokes equations and thermoelastic problems, will be discussed in the future.

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TECHNIQUE HYBRIDE DE TRANSFORMEE DE LAPLACE POUR LES PROBLEMES THERMIQUES VARIABLES ET NON LINEAIRE

Résumé—Une méthode numérique hybride combinant l'application de la transformation de Laplace et la méthode des différences finies (FDM) ou la méthode des éléments finis (FEM) est présentée pour des problèmes thermiques variables et non linéaires. Le domaine spatial des équations est discrétisé par FDM ou FEM et les termes non linéaires sont linéarisés par un développement en série de Taylor. Les termes dépendant du temps sont otés des équations linéarisées par la transformation de Laplace et ainsi les résultats à un instant donné peuvent être calculés sans un calcul pas-à-pas dans le domaine temporel. Quelques problèmes thermiques variables, monodimensionnels, non linéaires sont étudiés pour montrer l'efficacité et la précision de la présente méthode.

EINE HYBRIDE LAPLACE-TRANSFORMATION ZUR BERECHNUNG NICHTLINEARER TRANSIENTER THERMISCHER PROBLEME

Zusammenfassung—Für nichtlineare transiente thermische Probleme wird eine hybride numerische Methode vorgestellt, bei welcher die Laplace-Transformation mit dem Verfahren der Finiten-Differenzen (FDM) oder der Finiten-Elemente (FEM) kombiniert wird. Der räumliche Teil der beschreibenden Differentialgleichung wird durch FDM oder FEM diskretisiert, die nichtlinearen Terme werden mit Hilfe einer Taylor-Entwicklung linearisiert. Die zeitabhängigen Terme in der linearisierten Gleichung werden durch Laplace-Transformation auf konstante Terme zurückgeführt, so daß die Temperaturbeurteilung zu einer bestimmten Zeit ohne schrittweise Berechnungen im Zeitbereich ermittelt werden kann. Zur Demonstration der Effizienz und Genauigkeit des Verfahrens werden verschiedene eindimensionale nicht lineare transiente thermische Vorgänge untersucht.

ГИБРИДНЫЙ МЕТОД ПРЕОБРАЗОВАНИЙ ЛАПЛАСА ДЛЯ РЕШЕНИЯ НЕЛИНЕЙНЫХ НЕСТАЦИОНАРНЫХ ТЕПЛОВЫХ ЗАДАЧ

Аннотация—Для решения нелинейных нестационарных тепловых задач предложен гибридный численный метод, сочетающий метод преобразований Лапласа и метод конечных разностей (МКР) или конечных элементов (МКЭ). Пространственная область для определяющего уравнения дискретизируется в МКР или МКЭ, а нелинейные слагаемые линеаризуются разложением в ряд Тэйлора. К нестационарным слагаемым из линеаризованных уравнений применяется преобразование Лапласа, и таким образом могут быть получены результаты в конкретный момент времени без поэтапных вычислений во временной области. Для иллюстрации эффективности и точности предложенного метода исследуются несколько одномерных нелинейных нестационарных тепловых задач.